

# Entanglement of Multi-qudit States Constructed by Linearly Independent Coherent States: Balanced Case

G. Najarbashi <sup>\*</sup>, S. Mirzaei <sup>†</sup>

Department of Physics, University of Mohaghegh Ardabili, P.O. Box 179, Ardabil, Iran.

April 7, 2015

## Abstract

Multi-mode entangled coherent states are important resources for linear optics quantum computation and teleportation. Here we introduce the generalized balanced N-mode coherent states which recast in the multi-qudit case. The necessary and sufficient condition for bi-separability of such balanced N-mode coherent states is found. We particularly focus on pure and mixed multi-qubit and multi-qutrit like states and examine the degree of bipartite as well as tripartite entanglement using the concurrence measure. Unlike the N-qubit case, it is shown that there are qutrit states violating monogamy inequality. Using parity, displacement operator and beam splitters, we will propose a scheme for generating balanced N-mode entangled coherent states for even number of terms in superposition.

---

<sup>\*</sup>E-mail: Najarbashi@uma.ac.ir

<sup>†</sup>E-mail: SMirzaei@uma.ac.ir

# 1 Introduction

Coherent states, originally introduced by Schrodinger in 1926 [1], refer to a special kind of pure quantum-mechanical state of the light field corresponding to a single resonator mode which describe closest quantum state to a classical sinusoidal wave such as a continuous laser wave. However, for multi-mode fields, the prospects for nonclassical effects become even richer as long as the field states are not merely product states of each of the modes which means that the multi-mode is entangled state [2].

Entangled coherent states have many applications in quantum optics and quantum information processing [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Communication via entangled coherent quantum network is investigated in [13] where it is shown that the probability of performing successful teleportation through this network depends on its size. The nonlinear Mach-Zehnder interferometer is presented as a device whereby a pair of coherent states can be transformed into an entangled superposition of coherent states for which the notion of entanglement is generalized to include nonorthogonal, but distinct, component states [6, 14, 15]. In [16], it is proposed a scheme for generating multipartite entangled coherent states via entanglement swapping, with an example of a physical realization in ion traps and then bipartite entanglement of these multipartite states is quantified by the concurrence. The required conditions for the maximal entanglement in superposed bosonic coherent states of the form

$$|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle, \quad (1.1)$$

have been studied in references [17, 18], and subsequently have been generalized to the state

$$|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle, \quad (1.2)$$

in Ref. [19]. In Ref. [20] the generation of multipartite entangled  $SU(k+1)$  coherent states using a quantum network involving a sequence of k beam splitters have been investigated. Based on Glauber coherent states, the even and odd three-mode Schrodinger cat states and limita-

tions to sharing quantum correlations known as monogamy relations have been investigated in Ref. [21].

In this paper we consider the generalized balanced N-mode Glauber coherent states of the form

$$|\Psi_N^{(d)}\rangle = \frac{1}{\sqrt{M_N^{(d)}}} \sum_{i=0}^{d-1} \mu_i \underbrace{|\alpha_i\rangle \cdots |\alpha_i\rangle}_{N \text{ modes}}, \quad (1.3)$$

which is a general form of the balanced two-mode entangled coherent state  $|\psi\rangle_{bal} = \frac{1}{\sqrt{N}}(|\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle)$ . As two coherent states are in general nonorthogonal, they span a two dimensional qubit like Hilbert space  $\{|0\rangle, |1\rangle\}$ . Therefore, balanced two-mode coherent state  $|\psi\rangle_{bal}$  can be recast in two qubit form  $\sum_{i,j=0,1} a_{ij}|ij\rangle$ . The same argument can be formulated for other two modes coherent states such as

$$|\psi'\rangle_{bal} = \frac{1}{\sqrt{N'}}(|\alpha\rangle|\alpha\rangle + |\beta\rangle|\beta\rangle + |\gamma\rangle|\gamma\rangle), \quad (1.4)$$

if the set  $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$  are linearly independent, i.e. when they span the three dimensional qutrit like Hilbert space  $\{|0\rangle, |1\rangle, |2\rangle\}$ . Hence  $|\psi'\rangle_{bal}$  can be recast in two qutrit form  $|\psi'\rangle_{bal} = \sum_{i,j=0}^2 a_{ij}|ij\rangle$ . One may proceed in the same manner and represent the generalized balanced N-mode Glauber coherent states  $|\Psi_N^{(d)}\rangle$  as the multipartite qudit state

$$|\Psi_N^{(d)}\rangle = \sum_{i_0 i_1 \dots i_{d-1}=0}^{d-1} a_{i_0 i_1 \dots i_{d-1}} |i_0 i_1 \dots i_{d-1}\rangle, \quad (1.5)$$

if the set  $\{|\alpha_i\rangle\}_{i=0}^{d-1}$  are linearly independent. The entanglement of the state  $|\Psi_N^{(d)}\rangle$  can be calculated, using some useful measure such as concurrence, in the above multipartite qudit form.

The outline of this paper is as follows: In section 2 we begin with a rather simple case, i.e. multi-mode qubit like coherent states. Using concurrence measure we find the necessary and sufficient condition for maximally bipartite entanglement of these states. Furthermore, mixed states and monogamy inequality is investigated in this section. Section 3 devoted to the pure and mixed multi-qutrit case. It is shown that there are qutrit like states in which

the monogamy inequality is violated. The necessary and sufficient condition for separability of generalized balanced multi-mode coherent states is found in section 4. In section 5, we propose a scheme to produce the generalized balanced N-mode coherent states with superposition of even terms. Concluding remarks close this paper.

## 2 Multi-qubit case

We begin our discussion by recapitulating some fundamental notions of coherent states , as formulated in the traditional language of creation and annihilation operators  $\hat{a}$  and  $\hat{a}^\dagger$  as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle, \quad (2.6)$$

where  $\hat{D}(\alpha)$  called displacement operator and  $\alpha$  is an arbitrary complex number. Coherent state (also called Glauber state) refer to a special kind of pure quantum-mechanical state of the light field corresponding to a single resonator mode which describe closest quantum state to a classical sinusoidal wave such as a continuous laser wave. Alternatively, the coherent states are eigenstates of annihilation operators, i.e.  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ . The displacement operator is unitary with  $\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$  and multiplication rule

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{i\text{Im}(\alpha\beta^*)}\hat{D}(\alpha + \beta). \quad (2.7)$$

Expanding the exponential in displacement operator and using the relations  $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$  and  $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$  one can write the coherent states in terms of Fock states  $|n\rangle$  as

$$|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (2.8)$$

On the other hand, using, the orthonormality of Fock states ( $\langle n|m\rangle = \delta_{nm}$ ), the overlap of two coherent states reads

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta)}. \quad (2.9)$$

Let us now consider an N-mode state  $|\Psi_N^{(2)}\rangle$  expressed in terms of coherent states  $|\alpha\rangle$  and  $|\beta\rangle$  as follows

$$|\Psi_N^{(2)}\rangle = \frac{1}{\sqrt{M_N^{(2)}}}(|\alpha\rangle|\alpha\rangle \cdots |\alpha\rangle + \mu|\beta\rangle|\beta\rangle \cdots |\beta\rangle), \quad (2.10)$$

where  $\mu$ ,  $\alpha$  and  $\beta$  are generally complex numbers and  $M_N$  is a factor normalizing  $|\Psi_N^{(2)}\rangle$ , i.e.

$$M_N^{(2)} = 1 + |\mu|^2 + 2\text{Re}(\mu p^N), \quad (2.11)$$

in which  $p = \langle\alpha|\beta\rangle$  is equal to Eq. (2.9). Note that we used the superscript (2) for qubit-like states to distinguish it from that of qutrit like or other states in the next section. Two non-orthogonal coherent states  $|\alpha\rangle$  and  $|\beta\rangle$  are assumed to be linearly independent and span a two-dimensional subspace of the Hilbert space. The state  $|\Psi_N^{(2)}\rangle$  is in general an entangled state. To see this, all that we have to do is to evaluate its concurrence [22, 23, 24]. Recall that a general form of bipartite quantum state in the usual orthonormal basis  $|e_i\rangle$  is

$$|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{ij} |e_i \otimes e_j\rangle, \quad (2.12)$$

where  $d_1$  and  $d_2$  are dimensions of first and second part respectively. The norm of concurrence vector is defined as

$$C = \sqrt{\sum_{a=1}^{d_1(d_1-1)/2} \sum_{b=1}^{d_2(d_2-1)/2} |C_{ab}|^2}, \quad (2.13)$$

where  $C_{ab} = \langle\psi|\tilde{\psi}_{ab}\rangle$ ,  $|\tilde{\psi}_{ab}\rangle = (L_a \otimes L_b)|\psi^*\rangle$ , and  $L_a$  and  $L_b$  are the generators of  $SO(d_1)$  and  $SO(d_2)$  respectively. Note that  $|\psi^*\rangle$  is complex conjugate of  $|\psi\rangle$ . The concurrence in terms of coefficients  $a_{ij}$  is

$$C = 2 \sqrt{\sum_{i<j}^{d_1} \sum_{k<l}^{d_2} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}. \quad (2.14)$$

Moreover, when the state (2.12) is maximally entangled then  $C$  takes its maximum value  $\sqrt{2(d-1)/d}$  with  $d = \min\{d_1, d_2\}$ . Returning to the particular problem at hand, one can transform the  $|\Psi_N^{(2)}\rangle$  to a form analogous to Eq. (2.12) by defining the orthonormal basis

$$|0\rangle = |\alpha\rangle, \quad |1\rangle = \frac{|\beta\rangle - p|\alpha\rangle}{N_1}, \quad (2.15)$$

where  $N_1 = \sqrt{1 - |p|^2}$ . As the state  $|\Psi_N^{(2)}\rangle$  is symmetric, i.e. its form is unchanged by permuting each two modes, thus, without loss of generality, we can assume the first  $m$  modes as part one and the remaining  $(N - m)$  modes as the second part which implies that it can be represented in a simple two-qubit like form

$$\begin{aligned} |\Psi_N^{(2)}\rangle = & \frac{1}{\sqrt{M_N^{(2)}}} [(1 + \mu p^N) |\mathbf{00}'\rangle + \mu p^m (\sqrt{1 - |p|^{2(N-m)}}) |\mathbf{01}'\rangle + \mu p^{N-m} (\sqrt{1 - |p|^{2m}}) |\mathbf{10}'\rangle \\ & + \mu \sqrt{(1 - |p|^{2m})(1 - |p|^{2(N-m)})} |\mathbf{11}'\rangle], \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} |\mathbf{0}\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_m, & |\mathbf{0}'\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{N-m}, \\ |\mathbf{1}\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_m, & |\mathbf{1}'\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{N-m}. \end{aligned} \quad (2.17)$$

As for general two-qubit pure state  $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$  the concurrence (2.14) reduces to  $C = 2|ad - bc|$ , one can immediately deduce that Eq. (2.10) has concurrence

$$C_{m,N-m}^{(2)} = \frac{2|\mu|(1 - |p|^{2m})(1 - |p|^{2(N-m)})^{\frac{1}{2}}}{1 + |\mu|^2 + 2R|\mu|\cos(\phi + A)}, \quad (2.18)$$

where

$$A = N|\alpha||\beta|\sin(\theta_2 - \theta_1), \quad R = \exp\left[\frac{-N}{2}(|\alpha|^2 + |\beta|^2 - 2|\alpha||\beta|\cos(\theta_2 - \theta_1))\right], \quad (2.19)$$

in which we have defined  $\mu = |\mu|e^{i\phi}$ ,  $\alpha = |\alpha|e^{i\theta_1}$  and  $\beta = |\beta|e^{i\theta_2}$ . We used the superscript (2) for concurrence of qubit-like states to distinguish it from that of qutrit like or other states in the next section. Before proceeding with explicit examples, let us pause to make some preliminary remarks on concurrence (2.18). The multi-mode (2.10) is separable if and only if  $\mu = 0$  or  $p = 1$ . The former is trivial and the latter implies that the state  $|\alpha\rangle$ , is equal to  $|\beta\rangle$ , up to a phase factor, which is forbidden by our early assumption, i.e. linearly independent of  $|\alpha\rangle$  and  $|\beta\rangle$ . On the other hand, maximal entangled states can be obtained by solving the equation  $C = 1$  or equivalently the equation  $|\mu|^2 + 2|\mu|(R\cos(\phi + A) + R - 1) + 1 = 0$  must be solve for

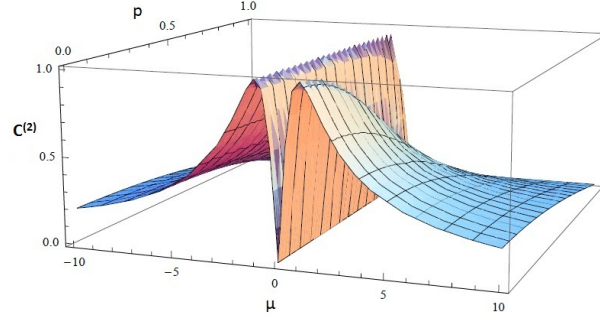


Figure 1: Concurrence of state  $|\Psi_N^{(2)}\rangle$  as a function of  $\mu$  and  $p$

$|\mu|$ . This in turn requires that  $N = 2m$ . Since this equation has negative discriminant,  $\mu, \alpha$  and  $\beta$  must be real numbers.

Now, let us suppose that  $\alpha, \beta$  and  $\mu$  are real numbers and for simplicity we consider two-mode state  $|\Psi_2^{(2)}\rangle = \frac{1}{\sqrt{M_N^{(2)}}}(|\alpha\rangle|\alpha\rangle + \mu|\beta\rangle|\beta\rangle)$  with concurrence

$$C_{1,1}^{(2)} = \frac{2|\mu(1-p^2)|}{|1+\mu^2+2\mu p^2|}, \quad (2.20)$$

which has maximum (i.e.  $C = 1$ ) in  $\mu = -1$ , with  $0 \leq p < 1$  and  $\mu = 1$ , with  $p = 0$  (see figure 1). Figure 2 indicates the behaviour of concurrence for different values of  $p = 0.1, 0.3$  and  $0.5$  which shows that for  $\mu > 0$  the concurrence significantly decreases by increasing  $p$ , while for  $\mu < 0$  the changes are insignificant. A profile of concurrence is depicted in figure 3 as a function of  $p$  in range  $[0, 1]$  for different values of positive  $\mu$ . As  $\mu$  is increased, the concurrence grows to attain its maximum value  $C = 1$  when  $p$  tend to zero namely  $|\alpha\rangle$  and  $|\beta\rangle$  are orthogonal ( $\alpha$  or  $\beta \rightarrow \infty$ ). It is instructive to compare the concurrence of the states  $|\Psi_N^{(2)}\rangle, |\Psi_N^{(3)}\rangle, |\Psi_N^{(4)}\rangle$  and  $|\Psi_N^{(5)}\rangle$  with fixed  $m = 1$  but several different values  $N = 2, 3, 4$  and  $5$  respectively (see figure 4). We found that the concurrence is more sensitive for positive  $\mu$  than for negative one. Similarly, one may fix  $N$  and  $p$ , (e.g. ten modes with  $p = 0.8$  in or example) and explore the behaviour of concurrence as a function of  $\mu$ . There is significant enhancement of the concurrence by the increasing  $m(\leq \frac{N}{2})$  where the maximum is achieved for balanced partition, i.e.  $m = \frac{N}{2}$  (see figure 5).

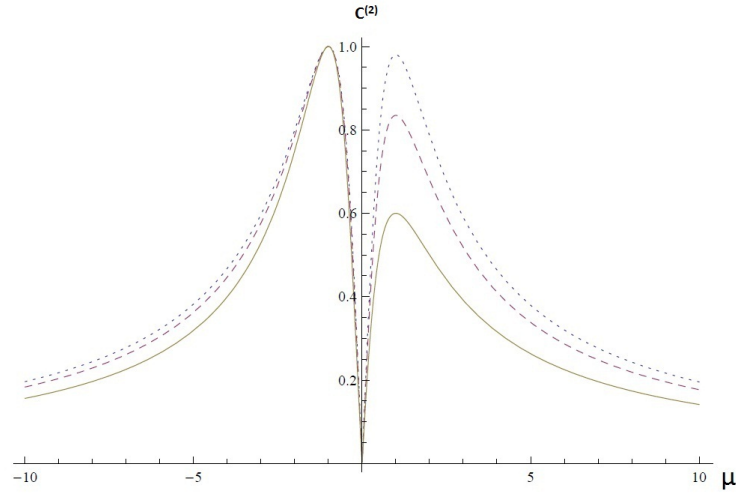


Figure 2: Concurrence  $C_{1,1}^{(2)}$  as a function of  $\mu$  for different values:  $p_1 = 0.1$  (dotted line),  $p_2 = 0.3$  (dashed line) and  $p_3 = 0.5$  (full line).

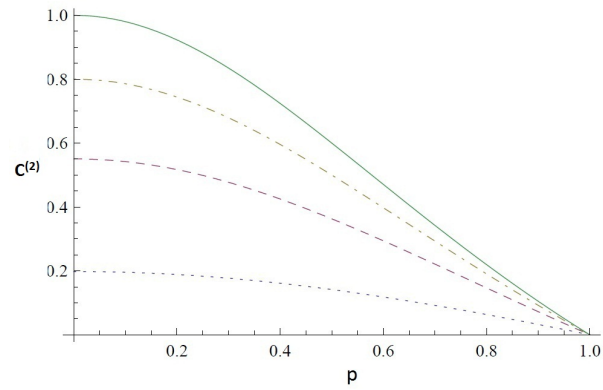


Figure 3: Concurrence  $C_{1,1}^{(2)}$  as a function of  $p$ , for different values of  $\mu = 0.1$  (dotted line),  $\mu = 0.3$  (dashed line),  $\mu = 0.5$  (dot-dashed line) and  $\mu = 1$  (full line).



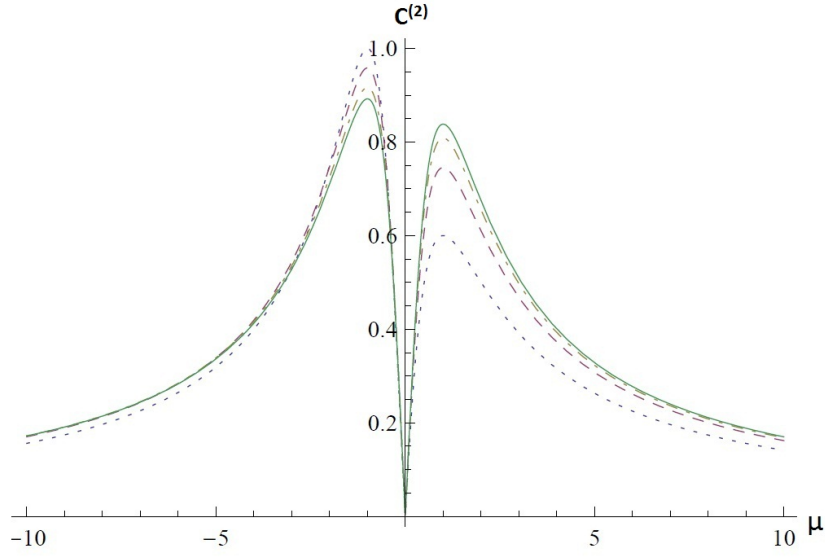


Figure 4: Concurrences  $C_{1,1}^{(2)}$  (dotted line),  $C_{1,2}^{(2)}$  (dashed line),  $C_{1,3}^{(2)}$  (dot-dashed line),  $C_{1,4}^{(2)}$  (full line) as a function of  $\mu$  in the range  $[-10, 10]$ .

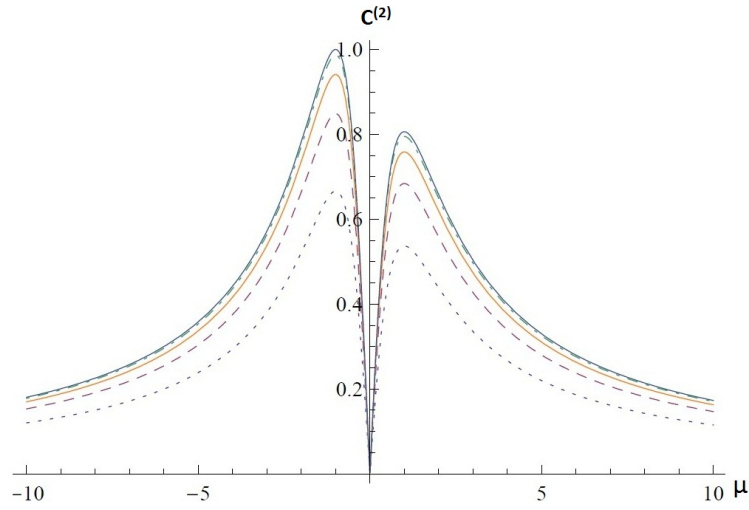


Figure 5: Concurrences  $C_{1,9}^{(2)}$  (dotted line),  $C_{2,8}^{(2)}$  (dashed line),  $C_{3,7}^{(2)}$  (orange line),  $C_{4,6}^{(2)}$  (dot-dashed line),  $C_{5,5}^{(2)}$  (full line) as a function of  $\mu$  in the range  $[-10, 10]$  for fixed number of mode  $N = 10$  and given  $p = 0.8$ .

## 2.1 Mixed states and monogamy inequality for multi-qubit case

Returning to the prototype multi-mode state (2.10) discussed earlier, let us focus on entanglement of mixed states that arise from (2.10) by partially tracing out some subsystem. To this end, let us consider the state (2.10) as three partitions  $A, B$  and  $D$  including  $m_1, m_2$  and  $m_3 = N - m_1 - m_2$  modes respectively, viz

$$\begin{aligned} |\Psi_N^{(2)}\rangle_{ABD} = \frac{1}{\sqrt{M_N^{(2)}}} & [(1 + \mu p^N) |\mathbf{00}'\mathbf{0}''\rangle + \mu N_1'' p^{m_1+m_2} |\mathbf{00}'\mathbf{1}''\rangle + \mu N_1' p^{N-m_2} |\mathbf{01}'\mathbf{0}''\rangle + \mu N_1 p^{N-m_1} |\mathbf{10}'\mathbf{0}''\rangle \\ & + \mu N_1' N_1'' p^{m_1} |\mathbf{01}'\mathbf{1}''\rangle + \mu N_1 N_1'' p^{m_2} |\mathbf{10}'\mathbf{1}''\rangle + \mu N_1 N_1' p^{N-m_1-m_2} |\mathbf{11}'\mathbf{0}''\rangle + \mu N_1 N_1' N_1'' |\mathbf{11}'\mathbf{1}''\rangle], \end{aligned} \quad (2.21)$$

where  $N_1 = \sqrt{1 - p^{2m_1}}$ ,  $N_1' = \sqrt{1 - p^{2m_2}}$ ,  $N_1'' = \sqrt{1 - p^{2m_3}}$  and

$$\begin{aligned} |\mathbf{0}\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_1}, & |\mathbf{0}'\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_2}, & |\mathbf{0}''\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_3}, \\ |\mathbf{1}\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_1}, & |\mathbf{1}'\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_2}, & |\mathbf{1}''\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_3}. \end{aligned} \quad (2.22)$$

With this decomposition at hand, we can obtain the reduced density matrices  $\rho_{AB} = \text{Tr}_D(|\Psi_N^{(2)}\rangle_{ABD}\langle\Psi_N^{(2)}|)$  and  $\rho_{AD} = \text{Tr}_B(|\Psi_N^{(2)}\rangle_{ABD}\langle\Psi_N^{(2)}|)$  by tracing out subsystems  $D$  and  $B$  respectively. In general the density matrices  $\rho_{AB}$  and  $\rho_{AD}$  are mixed. For any two-qubit mixed state, concurrence is defined as  $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$  (concentrating on two-qubit basis we suppress the superscript (2) throughout) where the  $\lambda_i$ 's are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix  $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ , with  $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$  in which  $\rho^*$  is the complex conjugate of  $\rho$  when it is expressed in a standard basis and  $\sigma_y$  represents the usual second Pauli matrix in a local basis  $\{|0\rangle, |1\rangle\}$  [22]. We can fix  $N$ ,  $m_1$  and  $m_2$  (e.g.  $N = 10$ ,  $m_1 = 2$  and  $m_2 = 3$ ) and explore monogamy inequality [25, 26, 27]

$$C_{A(BD)}^2 \geq C_{AB}^2 + C_{AD}^2, \quad (2.23)$$

where  $C_{AB}$  and  $C_{AD}$  are the concurrences of the reduced density matrices of  $\rho_{AB}$  and  $\rho_{AD}$  respectively and  $C_{A(BD)}$  is the concurrence of pure state  $|\Psi_N^{(2)}\rangle_{ABD}$  with respect to the partitions

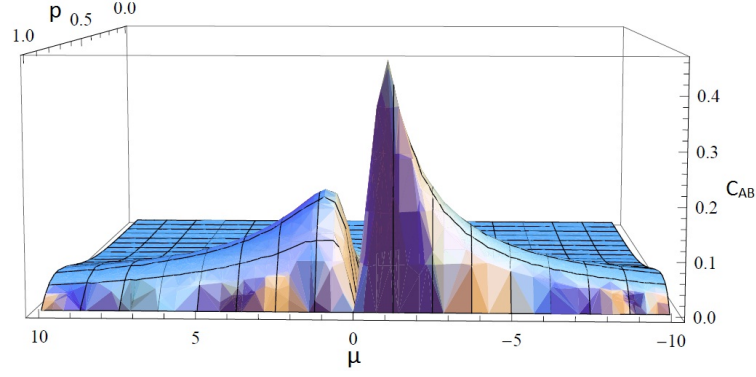


Figure 6: Concurrences  $C_{AB}$  as a function of  $\mu, p$  for fixed number of mode  $N = 10$ ,  $m_1 = 2$  and  $m_2 = 3$ .

$A$  and  $BD$ . The reduced density matrix  $\rho_{AB}$  has the form

$$\rho_{AB} = \frac{1}{M_N^{(2)}} \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{pmatrix}, \quad (2.24)$$

with

$$\begin{aligned} \rho_{11} &= 1 + 2\mu p^N + \mu^2 p^{2(m_1+m_2)}, & \rho_{12} &= \mu N'_1 (p^{N-m_2} + \mu p^{2m_1+m_2}), \\ \rho_{13} &= \mu N_1 (p^{N-m_1} + \mu p^{m_1+2m_2}), & \rho_{14} &= \mu N_1 N'_1 (p^{m_3} + \mu p^{m_1+m_2}), \\ \rho_{22} &= \mu^2 N_1'^2 p^{2m_1}, & \rho_{23} &= \mu^2 N_1 N'_1 p^{m_1+m_2}, & \rho_{24} &= \mu^2 N_1 N_1'^2 p^{m_1}, \\ \rho_{33} &= \mu^2 N_1^2 p^{2m_2}, & \rho_{34} &= \mu^2 N_1^2 N'_1 p^{m_2}, & \rho_{44} &= \mu^2 N_1^2 N_1'^2. \end{aligned} \quad (2.25)$$

The matrix  $R$ , associated to  $\rho_{AB}$ , has two nonzero eigenvalues  $\lambda_1, \lambda_2$  (i.e.  $\lambda_3 = \lambda_4 = 0$ ) where  $\lambda_1 > \lambda_2$  if  $\mu < 0$  and  $\lambda_2 > \lambda_1$  if  $\mu > 0$ , whence

$$C_{AB} = \max\{0, |\lambda_1 - \lambda_2|\} \quad \text{for all } \mu. \quad (2.26)$$

The concurrence  $C_{AB}$  can be represented diagrammatically as a function of  $0 \leq p < 1$  and  $\mu$  as shown in the figure 6. For negative  $\mu$  the maximum value  $C_{AB}$  occurs at  $\mu = -1$  when  $p \rightarrow 1$ , while for positive  $\mu$  the maximum occurs at  $\mu = 1$  and a point  $p < 1$ . Similar consequences hold

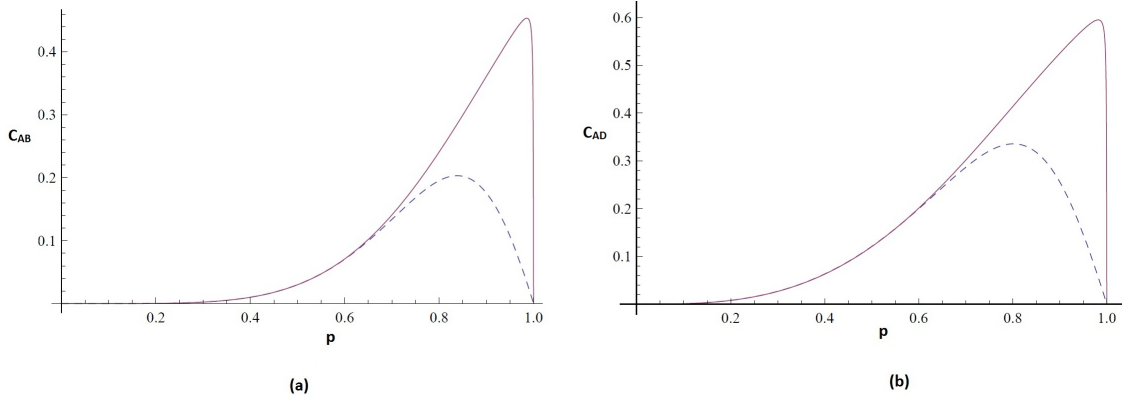


Figure 7: (a) Concurrences  $C_{AB}$  as a function of  $p$  for  $\mu = -1$  (full line) and  $\mu = 1$  (dashed line) ( for fixed number of modes  $N = 10$ ,  $m_1 = 2$  and  $m_2 = 3$ ). (b) The same profile for concurrence  $C_{AD}$ .

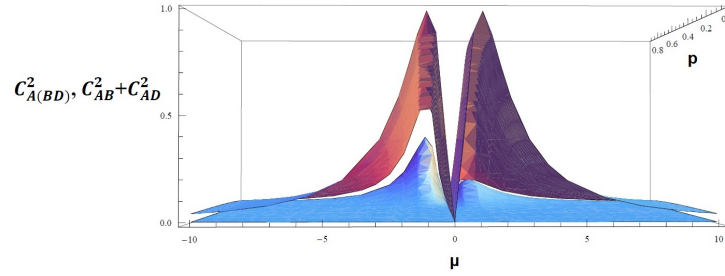


Figure 8:  $C^2_{A(BD)}$  (upper surface) and  $C^2_{AB} + C^2_{AD}$  (lower surface) as functions of  $\mu$  and  $p$  for fixed number of modes  $N = 10$ ,  $m_1 = 2$  and  $m_2 = 3$ .

for concurrence  $C_{AD}$  (see figure 7). On the other hand, if we define  $\tau_{ABD} = C^2_{A(BD)} - C^2_{AB} - C^2_{AD}$  then the positivity of  $\tau_{ABD}$  examine the monogamy inequality (2.23) (see figure 8). Figure 8 shows that the difference between the left and right-hand side of inequality (2.23) decrease by increasing  $p$  (see also figure 9). Later, in the next section we will revisit the monogamy inequality for qutrit states.

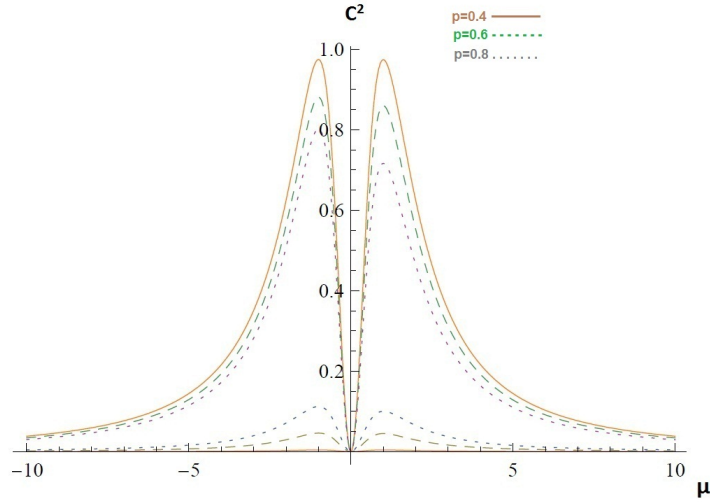


Figure 9: Concurrences as a function of  $\mu$  for fixed number of mode  $N = 10$ ,  $m_1 = 2$ ,  $m_2 = 3$  and  $p = 0.8$ .  $C_{AB}^2 + C_{AD}^2$  (dashed line) and  $C_{A(BD)}^2$  (full line)

### 3 Multi-qutrit case

It is tempting to add further terms to state (2.10) and explore its behavior from entanglement point of view. To this end, let us consider the N-mode state

$$|\Psi_N^{(3)}\rangle = \frac{1}{\sqrt{M_N^{(3)}}}(|\alpha\rangle \cdots |\alpha\rangle + \mu_1|\beta\rangle \cdots |\beta\rangle + \mu_2|\gamma\rangle \cdots |\gamma\rangle), \quad (3.27)$$

where  $p_1 = \langle\alpha|\beta\rangle$ ,  $p_2 = \langle\gamma|\beta\rangle$ ,  $p_3 = \langle\gamma|\alpha\rangle$  (for simplicity we assume that the all parameters are real) and

$$M_N^{(3)} = 1 + \mu_1^2 + \mu_2^2 + 2\mu_1 p_1^N + 2\mu_2 p_3^N + 2\mu_1 \mu_2 p_2^N. \quad (3.28)$$

We again assume that the three non-orthogonal coherent states  $|\alpha\rangle$ ,  $|\beta\rangle$  and  $|\gamma\rangle$  are linearly independent and span a three-dimensional subspace of the Hilbert space. Therefore we can define three orthonormal basis as

$$\begin{aligned} |0\rangle &= |\alpha\rangle, \\ |1\rangle &= \frac{1}{\sqrt{1-p_1^2}}(|\beta\rangle - p_1|\alpha\rangle), \\ |2\rangle &= \sqrt{\frac{1-p_1^2}{1-p_1^2-p_2^2-p_3^2+2p_1p_2p_3}} \left( |\gamma\rangle + \left(\frac{p_1p_3-p_2}{1-p_1^2}\right)|\beta\rangle + \left(\frac{p_1p_2-p_3}{1-p_1^2}\right)|\alpha\rangle \right). \end{aligned} \quad (3.29)$$

Once again, we can assume the first  $m$  ( $\leq \frac{N}{2}$ ) modes as part one and the remaining  $(N - m)$  modes as the second part and rewrite (3.27) as

$$\begin{aligned} |\Psi_N^{(3)}\rangle = & \frac{1}{\sqrt{M_N^{(3)}}} \left( (1 + \mu_1 p_1^N + \mu_2 p_3^N) |\mathbf{00}'\rangle + (\mu_1 N_1 p_1^{N-m} - \mu_2 x N_1 p_3^{N-m}) |\mathbf{10}'\rangle \right. \\ & + (\mu_1 N_1' p_1^m - \mu_2 x' N_1' p_3^m) |\mathbf{01}'\rangle + (\mu_1 N_1 N_1' - \mu_2 x x' N_1 N_1') |\mathbf{11}'\rangle - \mu_2 x' N_1' N_2 |\mathbf{21}'\rangle \\ & \left. - \mu_2 x N_1 N_2' |\mathbf{12}'\rangle + \mu_2 N_2 p_3^{N-m} |\mathbf{20}'\rangle + \mu_2 N_2' p_3^m |\mathbf{02}'\rangle + \mu_2 N_2 N_2' |\mathbf{22}'\rangle \right), \end{aligned} \quad (3.30)$$

in which the new basis is defined as usual qutrit like basis

$$\begin{aligned} |\mathbf{0}\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_m, & |\mathbf{0}'\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{N-m}, \\ |\mathbf{1}\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_m, & |\mathbf{1}'\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{N-m}, \\ |\mathbf{2}\rangle &\equiv \underbrace{|2\rangle \cdots |2\rangle}_m, & |\mathbf{2}'\rangle &\equiv \underbrace{|2\rangle \cdots |2\rangle}_{N-m}, \end{aligned} \quad (3.31)$$

and

$$\begin{aligned} x &= \frac{p_1^m p_3^m - p_2^m}{1 - p_1^{2m}}, & x' &= \frac{p_1^{N-m} p_3^{N-m} - p_2^{N-m}}{1 - p_1^{2(N-m)}}, \\ y &= \frac{p_1^m p_2^m - p_3^m}{1 - p_1^{2m}}, & y' &= \frac{p_1^{N-m} p_2^{N-m} - p_3^{N-m}}{1 - p_1^{2(N-m)}}, \\ N_1 &= \sqrt{1 - p_1^{2m}}, N_1' = \sqrt{1 - p_1^{2(N-m)}}, \\ N_2 &= \sqrt{1 - p_3^{2m} - x^2 N_1^2}, N_2' = \sqrt{1 - p_3^{2(N-m)} - x'^2 N_1'^2}, \end{aligned} \quad (3.32)$$

hence the  $|\Psi_N^{(3)}\rangle$  recast as two qutrit state. It is worthwhile to emphasize that the three parameters  $p_i$  ( $i=1,2,3$ ) are not, in general, independent of each other. This comes from the fact that both  $N_2^2$  and  $N_2'^2$ , being normalization factors, must be positive definite which impose the following constrains

$$\begin{cases} \lambda_- < p_3^{N-m} < \lambda_+, & \text{if } p_1^{2(N-m)} + p_2^{2(N-m)} \geq 1, \\ 0 < p_3^{N-m} < \lambda_+, & \text{if } p_1^{2(N-m)} + p_2^{2(N-m)} < 1, \end{cases} \quad (3.33)$$

where

$$\lambda_{\pm} = (p_1 p_2)^{N-m} \pm \sqrt{1 - p_1^{2(N-m)} - p_2^{2(N-m)} + (p_1 p_2)^{2(N-m)}}. \quad (3.34)$$

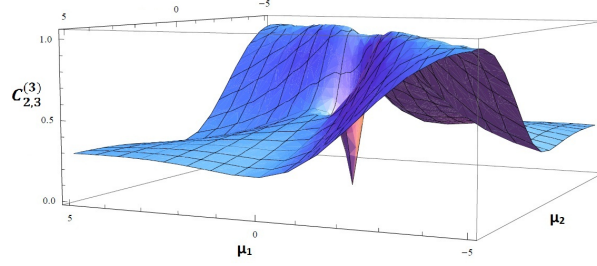


Figure 10: Concurrence  $C_{2,3}^{(3)}$  as a function of  $\mu_1$  and  $\mu_2$  for  $p_1 = 0.9$ ,  $p_2 = 0.89$  and  $p_3 = 0.8$ .

We can now use the concurrence formulae (2.14) to evaluate the entanglement of bipartite state  $|\varphi_N\rangle$ , i.e.

$$\begin{aligned}
 C_{m,N-m}^{(3)} = \frac{1}{M_N^{(3)}} & \left( |N_1 N_1' (\mu_1 + \mu_2 x x') + \mu_1 \mu_2 N_1 N_1' (x x' p_1^N + p_3^N + x' p_3^m p_1^{N-m} + x p_3^{N-m} p_1^m)|^2 \right. \\
 & + |N_1 N_2' (\mu_2 x + \mu_1 \mu_2 x p_1^N + \mu_1 \mu_2 p_3^m p_1^{N-m})|^2 + |N_1' N_2 (\mu_2 x' + \mu_1 \mu_2 x' p_1^N + \mu_1 \mu_2 p_3^{N-m} p_1^m)|^2 \\
 & + |\mu_2 N_2 N_2' (1 + \mu_1 p_1^N)|^2 + |\mu_1 \mu_2 N_1 N_1' N_2' y|^2 + |\mu_1 \mu_2 N_1' N_2 N_2' p_1^m|^2 + |\mu_1 \mu_2 N_1 N_1' N_2 y'|^2 \\
 & \left. + |\mu_1 \mu_2 N_1 N_2 N_2' p_1^{N-m}|^2 + |\mu_1 \mu_2 N_1 N_1' N_2 N_2'|^2 \right)^{\frac{1}{2}}.
 \end{aligned} \tag{3.35}$$

The concurrence  $C_{m,N-m}^{(3)}$  is reduced to  $C_{m,N-m}^{(2)}$ , when one of the  $\mu_1$  or  $\mu_2$  become zero. Let us first determine the separable states, i.e.  $C_{m,N-m}^{(3)} = 0$  which entail vanishing of all absolute terms appearing in  $C_{m,N-m}^{(3)}$ . Using the vanishing of final term implies that  $\mu_1$  or  $\mu_2$  must be zero. The forth term impose that  $\mu_2 = 0$  and subsequently the first term demands that  $\mu_1$  must also be zero. Hence the state  $|\Psi_N^{(3)}\rangle$  is bipartite separable, if and only if both  $\mu_1$  and  $\mu_2$  vanish. On the other hand, the state (3.30) has maximum concurrence, i.e.  $(C_{m,N-m}^{(3)})_{max} = \sqrt{\frac{4}{3}}$  if  $p_i = 0$  and  $\mu_{1,2} = \pm 1$  which means that it reduces to GHZ like states  $|\Psi_N^{(3)}\rangle_{GHZ} = \frac{1}{\sqrt{3}}(|00'\rangle \pm |11'\rangle \pm |22'\rangle)$ . We illustrate the behaviour of  $C_{2,3}^{(3)}$  as a function of  $\mu_1$  and  $\mu_2$  for given values  $p_i$  in figure 10. We note that the concurrence  $C_{2,3}^{(3)}$  is sensitive for the different values of the parameters. To see this, one may take  $p_2 = p_3 = 0$ , (i.e.  $\gamma \rightarrow \infty$ ) and  $p_1 \neq 0$

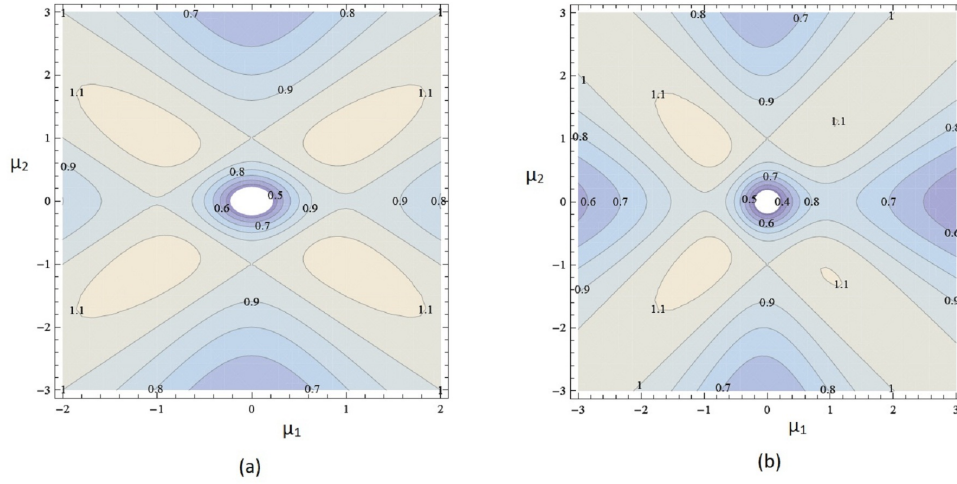


Figure 11: Concurrence  $C_{2,3}^{(3)}$  as a function of  $\mu_1$  and  $\mu_2$  for (a):  $p_1 = 0.3$  and (b):  $p_2 = 0.6$ .

that leads to the state

$$\begin{aligned}
 |\Psi_5^{(3)}\rangle = & \frac{1}{\sqrt{1+\mu_1^2+\mu_2^2+2\mu_1p_1^5}} ((1+\mu_1p_1^5)|00'\rangle + \mu_1N_1p_1^3|10'\rangle \\
 & + \mu_1N_1'p_1^2|01'\rangle + \mu_1N_1N_1'|11'\rangle + \mu_2|22'\rangle),
 \end{aligned} \tag{3.36}$$

with the following concurrence

$$\begin{aligned}
 C_{2,3}^{(3)} = & \frac{1}{1+\mu_1^2+\mu_2^2+2\mu_1p_1^5} (|\mu_1N_1N_1'|^2 + |\mu_2(1+\mu_1p_1^5)|^2 \\
 & + |\mu_1\mu_2N_1'p_1^2|^2 + |\mu_1\mu_2N_1p_1^3|^2 + |\mu_1\mu_2N_1N_1'|^2)^{\frac{1}{2}}.
 \end{aligned} \tag{3.37}$$

The behaviour of concurrence  $C_{2,3}^{(3)}$  as a function of  $\mu_1$  and  $\mu_2$  for given  $p_1$  is shown in figure 11.

### 3.1 Mixed states and monogamy inequality for multi-qutrit case

Now let us partition the state (3.27) to tripartite A,B and D including  $m_1$ ,  $m_2$  and  $m_3 = N - m_1 - m_2$  modes respectively. For the moment, suppose that  $\gamma \rightarrow \infty$ , i.e.  $p_2, p_3 = 0$  and



$p_1 \neq 0$ . Then the state (3.27) is reduced to three-qutrit state as

$$\begin{aligned} |\Psi_N^{(3)}\rangle_{ABD} = & \frac{1}{\sqrt{M_N^{(3)}}} ((1 + \mu_1 p_1^N) |\mathbf{00}'\mathbf{0}''\rangle + \mu_1 N_1'' p_1^{n+m} |\mathbf{00}'\mathbf{1}''\rangle + \mu_1 N_1' p_1^{N-m} |\mathbf{01}'\mathbf{0}''\rangle \\ & + \mu_1 N_1 p_1^{N-n} |\mathbf{10}'\mathbf{0}''\rangle + \mu_1 N_1' N_1'' p_1^n |\mathbf{01}'\mathbf{1}''\rangle + \mu_1 N_1 N_1' p_1^m |\mathbf{10}'\mathbf{1}''\rangle \\ & + \mu_1 N_1 N_1' p_1^{N-m-n} |\mathbf{11}'\mathbf{0}''\rangle + \mu_1 N_1 N_1' N_1'' |\mathbf{11}'\mathbf{1}''\rangle + \mu_2 |\mathbf{22}'\mathbf{2}''\rangle), \end{aligned} \quad (3.38)$$

where  $N_1 = \sqrt{1 - p_1^{2m_1}}$ ,  $N_1' = \sqrt{1 - p_1^{2m_2}}$ ,  $N_1'' = \sqrt{1 - p_1^{2m_3}}$  and

$$\begin{aligned} |\mathbf{0}\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_1}, & |\mathbf{0}'\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_2}, & |\mathbf{0}''\rangle &\equiv \underbrace{|0\rangle \cdots |0\rangle}_{m_3}, \\ |\mathbf{1}\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_1}, & |\mathbf{1}'\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_2}, & |\mathbf{1}''\rangle &\equiv \underbrace{|1\rangle \cdots |1\rangle}_{m_3}, \\ |\mathbf{2}\rangle &\equiv \underbrace{|2\rangle \cdots |2\rangle}_{m_1}, & |\mathbf{2}'\rangle &\equiv \underbrace{|2\rangle \cdots |2\rangle}_{m_2}, & |\mathbf{2}''\rangle &\equiv \underbrace{|2\rangle \cdots |2\rangle}_{m_3}. \end{aligned} \quad (3.39)$$

One can easily obtain the reduced density matrices  $\rho_{AB} = \text{Tr}_D(|\Psi_N^{(3)}\rangle_{ABD}\langle\Psi_N^{(3)}|)$  and  $\rho_{AD} = \text{Tr}_B(|\Psi_N^{(3)}\rangle_{ABD}\langle\Psi_N^{(3)}|)$  by partially tracing out subsystems  $D$  and  $B$  respectively. For general mixed states the concurrence is defined as  $|C|^2 = \sum_{\alpha\beta} |C_{\alpha\beta}|^2$  where  $C^{\alpha\beta} = \lambda_1^{\alpha\beta} - \sum_{i=2}^n \lambda_i^{\alpha\beta}$  with  $\lambda_1 = \max\{\lambda_i, i = 1, \dots, d^2\}$  and  $\lambda_i^{\alpha\beta}$  are the nonnegative square root of eigenvalues of  $\tau\tau^*$  defined as [28]

$$\tau^{\alpha\beta} \tau^{\alpha\beta*} = \sqrt{\rho}(E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \rho^* (E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta}) \sqrt{\rho}. \quad (3.40)$$

If we take  $N = 20, m_1 = 1, m_2 = 2, p = 0.7$  and  $\mu_2 = 0.4$ , then the components of concurrence vectors  $C_{AB}$  and  $C_{AD}$ , for all  $\mu_1$ , read

$$\begin{aligned} C_{AB} &= \left( \frac{|\mu_1|(0.0034 + 4.6 \times 10^{-6} \mu_1 + 0.003 \mu_1^2)}{1.35 + 0.004 \mu_1 + 2.32 \mu_1^2 + 0.0032 \mu_1^3 + \mu_1^4}, 0, 0, 0, 0, 0, \frac{0.38 \mu_1^2}{1.16 + 0.002 \mu_1 + \mu_1^2}, 0, 0 \right), \\ C_{AD} &= \left( \frac{|\mu_1|(0.81 + 0.001 \mu_1 + 0.7 \mu_1^2)}{1.35 + 0.004 \mu_1 + 2.32 \mu_1^2 + 0.0032 \mu_1^3 + \mu_1^4}, 0, 0, 0, 0, 0, -\frac{0.35 \mu_1}{1.16 + 0.002 \mu_1 + \mu_1^2}, 0, 0 \right). \end{aligned} \quad (3.41)$$

On the other hand, using (3.35), one finds that

$$C_{A(BD)}^2 = \frac{0.64 + 0.001 \mu_1 + 1.54 \mu_1^2}{1.16 + 0.002 \mu_1 + \mu_1^2}. \quad (3.42)$$

The behaviour of  $\tau_{ABD} = C_{A(BD)}^2 - C_{AB}^2 - C_{AD}^2$  as a function of  $\mu_1$  is shown in figure 12. One

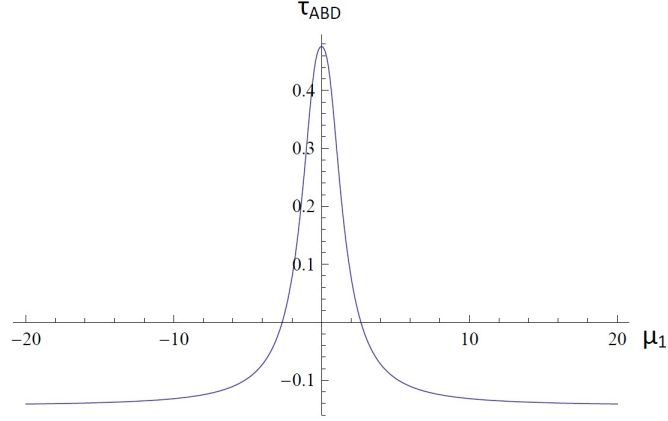


Figure 12:  $\tau_{ABD}$  as a function of  $\mu_1$  for given  $N = 20, m_1 = 1, m_2 = 2, p_1 = 0.7$  and  $\mu_2 = 0.4$ . For  $\mu_1 > 2.588$  we have a violation of monogamy inequality (2.23).

can immediately deduce that  $\tau_{ABD}$  becomes negative for  $\mu_1 > 2.588$  which is a violation of the inequality in Eq. (2.23). Clearly, for  $\mu_1 < 2.588$ , the monogamy inequality is satisfied. The other example which fulfils the monogamy inequality is the state  $\frac{1}{\sqrt{M}}(|\mathbf{00}'\mathbf{0}''\rangle + \mu_1|\mathbf{11}'\mathbf{1}''\rangle + \mu_2|\mathbf{22}'\mathbf{2}''\rangle)$  for which we have  $C_{AB} = C_{AD} = 0$  and  $C_{A(BD)} = \frac{2\sqrt{\mu_1^2 + \mu_2^2 + \mu_1^2\mu_2^2}}{1 + \mu_1^2 + \mu_2^2} \geq 0$ , hence  $\tau_{ABD} \geq 0$  for all  $\mu_1$  and  $\mu_2$ . The extremum inequality is accomplished for  $\mu_1, \mu_2 = \pm 1$ , namely  $C_{A(BD)}^2 = 4/3 \geq 0 = C_{AB}^2 + C_{AD}^2$ .

## 4 Separability of balanced N-mode coherent state

For completeness we mention that the separability of the aforementioned states, discussed above, refer to vanishing of all coefficients  $\mu_i$ , does not restrict to qubit and qutrit cases. To see this consider the following general balanced N-mode coherent state

$$|\Psi_N^{(d)}\rangle = \frac{1}{\sqrt{M_N^{(d)}}}(\mu_0|\alpha_0\rangle \cdots |\alpha_0\rangle + \mu_1|\alpha_1\rangle \cdots |\alpha_1\rangle + \cdots + \mu_{d-1}|\alpha_{d-1}\rangle \cdots |\alpha_{d-1}\rangle), \quad (4.43)$$

where without loss of generality we can take  $\mu_0 = 1$ . We wish to show that the above state is bipartite separable if and only if all  $\mu_i = 0$ , with  $i = 1, 2, \dots, d-1$ . Once again, to keep our discussion simple (yet generic in scope), let us focus attention on  $d = 3, 4$  cases.

**Case  $d = 3$ :** For qutrit states the concurrence is  $C_{m,N-m}^{(3)} = 2(\sum_{i < j} \sum_{k < l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2)^{\frac{1}{2}}$ . The term  $|a_{11}a_{22} - a_{12}a_{21}|$  is equal to last term  $|\mu_1\mu_2N_1N'_1N_2N'_2|$  in Eq. (3.35) which contains the product  $\mu_1\mu_2$ . Therefore, we have two cases:  $\mu_1 = 0, \mu_2 \neq 0$  or  $\mu_1 \neq 0, \mu_2 = 0$ , (the case both  $\mu_1 = \mu_2 = 0$  is trivial). The first case contradicts the vanishing of the term  $|a_{00}a_{22} - a_{02}a_{20}| = |\mu_2N_2N'_2(1 + \mu_1p_1^N)|$ , while the second case in contrast to vanishing of  $|a_{00}a_{11} - a_{01}a_{10}| = |N_1N'_1(\mu_1 + \mu_2xx') + \mu_1\mu_2N_1N'_1(xx'p_1^N + p_3^N + x'p_3^m p_1^{N-m} + xp_3^{N-m}p_1^m)|$ . Hence, the qutrit state  $|\Psi_N^{(3)}\rangle$  is bi-separable if and only if both  $\mu_1 = \mu_2 = 0$ .

**Case  $d = 4$ :** One step further is to consider the following state

$$|\Psi_N^{(4)}\rangle = \frac{1}{\sqrt{M_N^{(4)}}} |\alpha_0\rangle \cdots |\alpha_0\rangle + \mu_1 |\alpha_1\rangle \cdots |\alpha_1\rangle + \mu_2 |\alpha_2\rangle \cdots |\alpha_2\rangle + \mu_3 |\alpha_3\rangle \cdots |\alpha_3\rangle. \quad (4.44)$$

The last term of concurrence  $C_{m,N-m}^{(4)} = 2(\sum_{i < j} \sum_{k < l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2)^{\frac{1}{2}}$ , i.e.  $|a_{22}a_{33} - a_{23}a_{32}| \propto \mu_2\mu_3$  which is equal zero for all  $p_i$  if  $\mu_2 = 0, \mu_3 \neq 0$  or  $\mu_2 \neq 0, \mu_3 = 0$ . The former case contradicts the vanishing of the terms  $|a_{11}a_{33} - a_{13}a_{31}| \propto \mu_3(\mu_1 + \mu_2xx')$  and  $|a_{00}a_{33} - a_{03}a_{30}| \propto \mu_3(1 + \mu_1p_1^N + \mu_2p_3^N)$ , while the latter case contradicts the the vanishing of the terms

$$\begin{aligned} |a_{11}a_{22} - a_{12}a_{21}| &\propto \mu_1(\mu_2 + f_1\mu_3) + f_2\mu_2\mu_3, \\ |a_{00}a_{22} - a_{02}a_{20}| &\propto \mu_2 + f_3\mu_3 + f_4\mu_1\mu_2 + f_5\mu_1\mu_3 + f_6\mu_2\mu_3. \end{aligned} \quad (4.45)$$

Hence both  $\mu_2, \mu_3 = 0$ . On the other hand, the vanishing of the first term

$$|a_{00}a_{11} - a_{01}a_{10}| \propto \mu_1 + g_1\mu_2 + g_2\mu_3 + g_3\mu_1\mu_2 + g_4\mu_1\mu_3 + g_5\mu_2\mu_3, \quad (4.46)$$

imposes that  $\mu_1 = 0$ , where  $f_i$ 's and  $g_i$ 's are some functions of  $p_i$ 's.

**General case :** We next turn to the general discussion of the problem with arbitrary  $d$ . As should be clear from the discussion above, in each step the vanishing of the term  $|a_{ii}a_{jj} - a_{ij}a_{ji}|, i < j$  leads to two cases  $\mu_i = 0, \mu_j \neq 0$  or  $\mu_i \neq 0, \mu_j = 0$ . The former case contradicts the vanishing of the terms  $|a_{ii}a_{j'j'} - a_{ij'}a_{j'i}|, j' = 0, 1, \dots, j-2$ , while the latter case contradicts the vanishing of the terms  $|a_{i'i'}a_{jj} - a_{i'j}a_{ji'}|, i' = 0, 1, \dots, i-1$ .

## 5 Generation of balanced N-mode entangled coherent state: Even terms

In this section, we want to propose a scheme to produce the general balanced N-mode entangled coherent states. To this aim we first need to have the superposition of even and odd number of coherent states like as  $|\alpha\rangle + \mu|\beta\rangle$  and  $|\alpha\rangle + \mu_1|\beta\rangle + \mu_2|\gamma\rangle$ , (up to normalization factors), and so on. In general, the superpositions of coherent states are difficult to produce, and fundamentally this could be due to extreme sensitivity to environmental decoherence (see, e.g., [14, 15, 29, 30, 31, 32, 33]). Here we restrict ourselves to the even cases. One may use the displacement operator together with parity operation [34, 35], to construct the unitary operation

$$\hat{U}(\lambda, \alpha) = e^{i\lambda\hat{D}(\alpha)\hat{\Pi}}, \quad (5.47)$$

where  $\lambda$  is real number,  $\hat{\Pi} = \cos(\pi\hat{a}^\dagger\hat{a})$  is a Hermitian and unitary operator with property  $\hat{\Pi}|n\rangle = (-1)^n|n\rangle$ , and  $\hat{D}(\alpha)$  is usual displacement operator (2.6). Using the fact that  $[\hat{D}(\alpha)\hat{\Pi}]^2 = \hat{I}$ , it can be rewrite as

$$\hat{U}(\lambda, \alpha) = \cos \lambda \hat{I} + i \sin \lambda \hat{D}(\alpha) \hat{\Pi}. \quad (5.48)$$

The action of such an operator on a vacuum state is

$$\hat{U}(\lambda, \alpha)|0\rangle = \cos \lambda |0\rangle + i \sin \lambda |\alpha\rangle. \quad (5.49)$$

In order to obtain a linear combination of two arbitrary Glauber coherent states, it is enough to use another displacement operator,  $\hat{D}(\beta)$ :

$$\hat{D}(\beta)\hat{U}(\lambda, \alpha)|0\rangle = \cos \lambda |\beta\rangle + i \sin \lambda |\alpha + \beta\rangle. \quad (5.50)$$

Using  $\hat{V}(\alpha, \beta, \lambda) = \hat{D}(\alpha)\hat{U}(\lambda, \beta - \alpha)$ , one may recast the above state in a convenient form as follows

$$\hat{V}(\alpha, \beta, \lambda)|0\rangle = \cos \lambda |\alpha\rangle + i \sin \lambda e^{i\text{Im}(\alpha\beta^*)}|\beta\rangle. \quad (5.51)$$

The method reported here may be extended to the case of a superposition involving just  $2^N$  coherent states, by considering the following multiplication

$$\hat{V}^N = \prod_{k=1}^N \hat{V}(\lambda_k, \alpha_k, \beta_k). \quad (5.52)$$

We next use polarizing beam splitter (PBS). The polarizing beam splitter is commonly made by cementing together two birefringent materials like calcite or quartz, and has the property of splitting a light beam into its orthogonal linear polarizations. The beam splitter interaction given by the unitary transformation

$$\hat{B}_{i-1,i}(\theta) = \exp[\theta(\hat{a}_{i-1}^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_{i-1})]. \quad (5.53)$$

which  $\hat{a}_{i-1}$ ,  $\hat{a}_i$ ,  $\hat{a}_{i-1}^\dagger$  and  $\hat{a}_i^\dagger$  are the annihilation and creation operators of the field mode  $i-1$  and  $i$ , respectively. Using Baker-Hausdorf formula, the action of the beam splitter on two modes  $i-1$  and  $i$ , can be expressed as

$$\hat{B}_{i-1,i}(\theta) \begin{pmatrix} \hat{a}_{i-1} \\ \hat{a}_i \end{pmatrix} \hat{B}_{i-1,i}^\dagger(\theta) = \begin{pmatrix} \hat{a}'_{i-1} \\ \hat{a}'_i \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{a}_{i-1} \\ \hat{a}_i \end{pmatrix}. \quad (5.54)$$

We consider 50 – 50 beam splitter, i.e.  $\theta = \pi/4$  with the following operation

$$\hat{B}_{1,2}(\pi/4)|\alpha'\rangle_1|0\rangle_2 = |\frac{\alpha'}{\sqrt{2}}\rangle_1|\frac{\alpha'}{\sqrt{2}}\rangle_2. \quad (5.55)$$

This result says that like classical light wave where the incident intensity is evenly divided between the two output beams, e.g. half the incident average photon number,  $\frac{|\alpha|^2}{2}$ , emerges in each beam. Note that the output is not entangled. For producing entangled coherent state suppose that our input state be a superposition of two coherent states as  $|\alpha'\rangle_1 + \mu_1|\beta'\rangle_1$ .

Following the procedure above, we may then, obtain the output state as

$$\hat{B}_{1,2}(\pi/4)(|\alpha'\rangle_1 + \mu_1|\beta'\rangle_1) \otimes |0\rangle_2 = |\frac{\alpha'}{\sqrt{2}}\rangle_1|\frac{\alpha'}{\sqrt{2}}\rangle_2 + \mu_1|\frac{\beta'}{\sqrt{2}}\rangle_1|\frac{\beta'}{\sqrt{2}}\rangle_2. \quad (5.56)$$

By renaming  $\alpha \equiv \frac{\alpha'}{\sqrt{2}}$  and  $\beta \equiv \frac{\beta'}{\sqrt{2}}$  we have  $|\alpha\rangle_1|\alpha\rangle_2 + \mu_1|\beta\rangle_1|\beta\rangle_2$ , which is a two-mode entangled coherent state. In order to obtain the three-mode entangled coherent states we use two beam

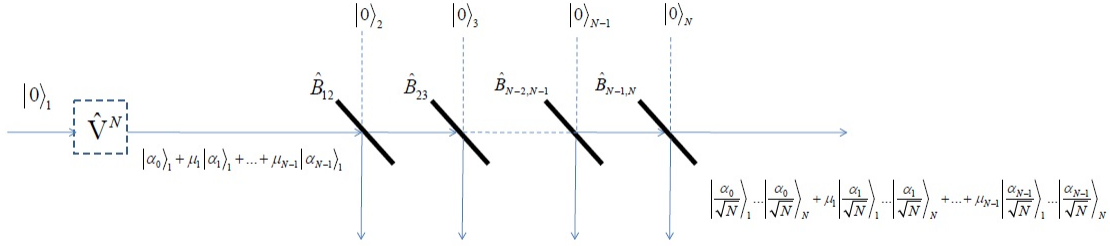


Figure 13: A protocol for generation of balanced N-mode entangled coherent states

splitters with reflectivity amplitude of  $\frac{1}{\sqrt{3}}$ , modes 1 and 2. At first beam splitter, the modes 1 and 2 are combined with reflectivity amplitude of  $\frac{1}{\sqrt{3}}$  and subsequently, the modes 2 and 3 undergo a 50-50 beam splitter, i.e.

$$\begin{aligned} & \hat{B}_{2,3}(\pi/4) \hat{B}_{1,2}(\cos^{-1}(\frac{1}{\sqrt{3}})) [(|\alpha'\rangle_1 + \mu_1 |\beta'\rangle_1) \otimes |0\rangle_2 \otimes |0\rangle_3] \\ &= |\frac{\alpha'}{\sqrt{3}}\rangle_1 |\frac{\alpha'}{\sqrt{3}}\rangle_2 |\frac{\alpha'}{\sqrt{3}}\rangle_3 + \mu_1 |\frac{\beta'}{\sqrt{3}}\rangle_1 |\frac{\beta'}{\sqrt{3}}\rangle_2 |\frac{\beta'}{\sqrt{3}}\rangle_3, \end{aligned} \quad (5.57)$$

which can be recasted in the  $|\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3 + \mu_1 |\beta\rangle_1 |\beta\rangle_2 |\beta\rangle_3$ . The procedure can be easily generalized to prepare N-mode ( $N = 2^k$ ) entangled coherent states. To this end, we should first apply the unitary operator  $\hat{V}^{2^k} = \prod_{i=1}^{2^k} \hat{V}(\lambda_i, \alpha_i, \beta_i)$  to the initial state  $|0\rangle_1$  and subsequently use the sequential beam splitters  $\hat{B}_{N-1,N} \hat{B}_{N-2,N-1} \dots \hat{B}_{2,3} \hat{B}_{1,2}$  with  $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{N-1}}, \frac{1}{\sqrt{N}}$  reflectivity amplitude respectively. The final result is a qudit like  $2^k$ -mode entangled coherent state (1.3) (see figure 13). The scheme to generate N-mode entangled coherent state with N odd is under debate. As mentioned above, the problem is to produce the superposition of arbitrary coherent states. For example a cat state  $|\alpha\rangle_{\pm} = |\alpha\rangle \pm |-\alpha\rangle$ , served as a qubit for quantum logical encoding, can be easily produce in a Kerr medium [7] while the problem of generating discrete superpositions of coherent states in the process of light propagation through a nonlinear Kerr medium, which is modelled by the anharmonic oscillator, is rather restricted [36, 37, 38]. But in general the problem, even in the rather simple case  $|\alpha\rangle + |\beta\rangle + |\gamma\rangle$ , is open under debate.

## 6 Conclusion

In summary, we have introduced generalized balanced N-mode entangled coherent states  $|\Psi_N^{(d)}\rangle$  which can be recast as multi-qudit pure states. To this aim we required linearly independent of various coherent states appearing in superposition. For simplicity, we stick here to pure and mixed states involving real parameters, except for pure N-qubit states. It was shown that the state  $|\Psi_N^{(d)}\rangle$  is separable if and only if all  $\mu_i = 0$ . To analyze the entanglement of pure and mixed qubit like states, we used the concurrence measure, introduced by Wootters, while for general pure N-mode cases we applied concurrence vector introduced by Akhtarshenas. For N-qubit pure states, the concurrence  $C_{m,N-m}^{(2)}$  shows a significant enhancement by increasing  $m \leq \frac{N}{2}$  where the maximum is achieved for balanced partition, i.e.  $m = \frac{N}{2}$ . The same result holds for qutrit states which means that the concurrence  $C_{m,N-m}^{(3)}$  has maximum value for  $p_i = 0$  and  $\mu_{1,2} = \pm 1$ , whence the state reduces to GHZ state. We saw that, unlike in N-qubit cases, there are pure N-qutrit states violating monogamy inequality. Based on parity, displacement operators and beam splitters, we have proposed a protocol for generating balanced N-mode entangled coherent states with even number of terms appearing in superposition.

### Acknowledgments

The authors also acknowledge the support from the University of Mohaghegh Ardabili.

## References

- [1] E. Schrödinger *Naturwissenschaften* **14**, 664 (1926).
- [2] A. Einstein, B. Podolski and N. Rosen, *Phys. Rev.* **47**, 777-780 (1935).
- [3] P. T. Cochrane, G. J. Milburn, and W. J. Munro, *Phys. Rev. A* **59**, 2631 (1999).
- [4] M. C. de Oliveira and W. J. Munro, *Phys. Rev. A* **61**, 042309 (2000).

- [5] H. Jeong and M. S. Kim, Phys. Rev. A **65**, 042305 (2002).
- [6] Barry C. Sanders, Phys. Rev. A **45**, 9 (1992).
- [7] T. C. Ralph, A. Gilchrist and G. J. Milburn, Phys. Rev. A **68**, 042319 (2003)
- [8] W. J. Munro, G. J. Milburn, and B. C. Sanders, Phys. Rev. A **62**, 052108 (2001).
- [9] X.Wang, Phys. Rev. A **64**, 022302 (2001).
- [10] S. J. van Enk, O. Hirota, Phys. Rev A **67**, 022313 (2001).
- [11] X. Wang, B. C. Sanders , S.H. Pan, J. Phys. A **33**, 7451 (2000).
- [12] W. Vogel and J. Sperling, Phys. Rev. A **89**, 052302 (2014).
- [13] A. El Allati, Y. Hassouni, N. Metwally, Phys. Scr, **83** 065002 (2011).
- [14] Y. W. Cheong, J. Lee, Journal of the Korean Physical Society, Vol. **51**, No. 4 (2007).
- [15] S. J. van Enk, Phys. Rev. Lett. , Vol. **91**, No. 1 (2003).
- [16] X. Wang, B. C. Sanders, Phys. Rev. A **65**, 012303 (2002).
- [17] X. Wang, J. Phys. A, Math. Gen. **35**(1), 165173 (2002).
- [18] H. Fu, X. Wang, A. I. Solomon, Phys. Lett. A **291**, 7376 (2001).
- [19] G. Najarbashi, Y. Maleki, Int J. Theor. Phys (2011).
- [20] M. Daoud, E. B. Choubabi, International Journal of Quantum Information, Vol. **10**, No. 1 (2012).
- [21] M. Daoud, R. Ahl Laamara, R. Essaber, W. Kaydi, Phys. Scr. **89**, 065004 (2014).
- [22] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).



- [23] W. K. Wootters, Quantum Information and Computation, Vol. 1, No. 1 (2001).
- [24] S. J. Akhtarshenas, J. Phys. A: Math. Gen. **38** 67776784 (2005).
- [25] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
- [26] J. S. Kim, A. Das and B. C. Sanders, Phys. Rev. A **79**, 012329 (2009).
- [27] B. C. Sanders, J. S. Kim, Applied Mathematics and Information Sciences **4(3)** (2010).
- [28] Y. Q. Li, G. Q. Zhu, Front. Phys. China, **3(3)**: 250257 (2008).
- [29] B. C. Sanders, J. Phys. A: Math. Theor. **45** (2012).
- [30] G. J. Milburn, Phys. Rev. A **33**, No. 1 (1986).
- [31] G. J. Milburn, C. Holmes, Phys. Rev. Lett., Vol. **56**, No. 21 (1986).
- [32] B. Yurke, D. Stoler, Phys. Rev. Lett., Vol. **57**, No. 1 (1986).
- [33] B. Yurke, D. Stoler, Phys. Rev. A Vol. **35**, No. 11 (1987).
- [34] A. Messina, B. Militello and A. Napoli, Proceedings of Institute of Mathematics of NAS of Ukraine, Vol. **50**, Part 2 (2004).
- [35] A. Messina, G. Draganescu, Preprint, arxiv: quant-ph/1306.2524v2 (2013).
- [36] M. Paprzycka, R. Tanas, Quantum opt, **4** (1992).
- [37] A. Miranowicz, R. Tanas, S. Kielich, Quantum opt, **2**, 253 (1990).
- [38] Ts. Gantsog, R. Tanas , Quantum opt, **3**, 33 (1991).